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Conservation Form in Computational Magnetohydrodynamics*

IRVIN R. LINDEMUTH

Lawrence Livermore Laboratory, Livermore, California 94550

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The question of whether or not to use a conservation form of the internal energy equation in computational magnetohydrodynamics is considered. Conservation errors resulting from both spatial and temporal difference methods are demonstrated, and means of eliminating the spatial errors and reducing the temporal ones are discussed. It is shown that only in the limit $\Delta t \to 0$ can a nonconservation form of the internal energy equation be used without destroying in the finite difference equations any of the conservation properties of the magnetohydrodynamic partial differential equations.

INTRODUCTION

A survey of recent plasma physics journals and conference proceedings [1] indicates that there is a renewed interest in the computer simulation of current plasma experiments by numerical solution of magnetohydrodynamic partial differential equations. This renewed interest is, of course, due on the one hand to the rapid advances in computer technology which have made practical multi-dimensional calculations which would have been prohibitively expensive, even if suitable numerical methods had been available, in the early sixties when the pioneering work of Hain *et al* [2]. on one-dimension occurred. On the other hand, there is an apparent realization that magnetohydrodynamic models, even in cases where their conditions of applicability are not rigorously satisfied, can and indeed do predict and lead to an increased understanding of the dynamics of many experiments.

The application of computational magnetohydrodynamics to the study of an experiment requires two steps. The first step is the selection of an appropriate physical model. Braginskii [3] has given a rather complete treatment of a two-component plasma, and most models for which numerical solutions have been

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obtained can be considered subsets of the model given there. Roberts and Potter [4] survey a variety of additional models.

The second step in the application of computational magnetohydrodynamics is the selection of a suitable finite difference approximation to the partial differential equations to be solved. The physical model will in general involve transport, both convective and diffusive, of physical quantities such as mass, momentum, energy, and magnetic flux. The fundamental numerical problems, particularly in Eulerian calculations, of numerical stability, diffusion, and dispersion are well documented and Richtmyer and Morton [5] give a complete summary. In general, the analysis of these problems is limited to linear differential equations. Additional nonphysical effects can be introduced when nonlinear equations are considered. For example, although magnetohydrodynamic equations conserve mass, momentum, and energy, these conservation properties are not necessarily retained in the finite difference approximation. "Conservative" difference approximations can be constructed, and the Lax–Wendroff [6] method, for example, has been used extensively in hydrodynamics. Roache [7] has suggested that the real significance of the original Lax [8] paper is not the actual difference method but the introduction of the "conservation form" difference equations which rigorously guarantee conservation of mass, momentum, and energy.

In magnetohydrodynamics, generalizations of the Lax–Wendroff method have been used by Freeman and Lane [9] and Potter [10]. However, in many cases of interest, the magnetic energy can be orders of magnitude larger than the thermal and kinetic energy of the plasma. Hence, as Roberts and Potter have indicated without elaboration, when total energy is conserved, small errors in the magnetic field energy can lead to large errors in the thermal energy. Contradicting the experience of the hydrodynamicists, Roberts and Potter suggest the use of a nonconservative difference equation for thermal energy transport.

The computational magnetohydrodynamicist is thus faced with a dilemma: whether or not to use a conservative energy transport equation. To resolve the dilemma, this paper examines finite difference approximations to both nonconservation and conservation forms of the internal energy equation. We show in detail the well-known result that the nonconservation equation can lead to a loss or gain of energy from the system considered. It is shown that the lack of conservation is due to both spatial and temporal differencing. On the other hand, we show in detail that an equation guaranteeing total energy conservation can lead to a nonphysical interchange of energy from one type to another. The nonphysical interchanges due to both spatial and temporal differencing are considered here. It is shown that the errors due to the spatial differencing can be eliminated quite easily and that those due to the temporal differencing can be minimized. Hence to a very satisfactory degree, conservative and nonconservative forms of the difference equations, as with the differential equations, can be equivalent.

PHYSICAL MODEL

A simple one-dimensional magnetohydrodynamic model will be sufficient to illustrate the concepts to be discussed below. The model uses a continuity equation, an equation of motion, an internal energy equation, and Faraday's law. The appropriate Eulerian differential equations as usually written are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{1}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}(\rho v v) + \frac{\partial p}{\partial x} + \frac{\partial B^2/2}{\partial x} = 0, \tag{2}$$

$$\frac{\partial(\rho \epsilon)}{\partial t} + \frac{\partial}{\partial x}(\rho \epsilon v) + p\frac{\partial v}{\partial x} - \eta\left(\frac{\partial B}{\partial x}\right)^2 = 0, \tag{3}$$

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x}(vB) - \frac{\partial}{\partial x}\left(\eta \frac{\partial B}{\partial x}\right) = 0. \tag{4}$$

In Eqs. (1)–(4) ρ is the mass density, v the fluid velocity, ϵ the specific internal energy, B the magnetic field, p the material pressure, and η the electrical resistivity. For completeness, the model requires equations of state relating p and η to ρ and ϵ.

A conservation equation is of course one which indicates that the time rate of change of a quantity within a volume is equal to the integral of a second quantity over the surface enclosing that volume. Equation (1), for example, gives

$$\int_{x_1}^{x_2} \frac{\partial \rho}{\partial t}\, dx = \frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho\, dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}(\rho v)\, dx = (\rho v)|_{x_1} - (\rho v)|_{x_2},$$

i.e., the time rate of change of mass within the volume is equal to the net mass flow rate into the volume. Similarly, Eq. (2) expresses conservation of momentum and Eq. (4) expresses conservation of magnetic flux. Equations (1) through (4) can be combined to give an equation of conservation of total energy:

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{1}{2}\rho v^2 + \frac{B^2}{2}\right) + \frac{\partial}{\partial x}\left(\rho\epsilon v + \frac{1}{2}\rho v^3 + pv + vB^2 - \eta B\frac{\partial B}{\partial x}\right) = 0. \tag{5}$$

The conservation equations have the general form

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0. \tag{6}$$

Basic Numerical Considerations

We consider only "cell-centered" finite difference approximations to Eqs. (1) to (5), so that each quantity including the velocity is defined at the same space–time point (a commonly used alternative is the definition of velocity at cell interfaces). The total mass, momentum, internal energy, magnetic flux, kinetic energy, magnetic energy, and total energy within the finite difference domain at time t^n are

$$M^n = \sum \rho_j{}^n \, \Delta x, \tag{7}$$

$$P^n = \sum \rho_j{}^n v_j{}^n \, \Delta x, \tag{8}$$

$$E_I{}^n = \sum \rho_j{}^n \epsilon_j{}^n \, \Delta x, \tag{9}$$

$$\Phi^n = \sum B_j{}^n \, \Delta x, \tag{10}$$

$$E_k{}^n = \sum \tfrac{1}{2} \rho_j{}^n v_j{}^n v_j{}^n \, \Delta x, \tag{11}$$

$$E_B{}^n = \sum \tfrac{1}{2} B_j{}^n B_j{}^n \, \Delta x, \tag{12}$$

$$E_T{}^n = E_I{}^n + E_k{}^n + E_B{}^n, \tag{13}$$

respectively, where $\rho_j{}^n = \rho(t^n, x_j)$, etc., and the summation is over all values of j; for our present purposes, boundary conditions which lead to "half-cell" interpretations can be ignored.

A typical finite difference approximation to (6) would have the form

$$(Q_j^{n+1} - Q_j{}^n)/\Delta t + [(F_+)_j - (F_-)_j]/\Delta x = 0. \tag{14}$$

If, in addition, $(F_+)_j = (F_-)_{j+1}$, Eq. (14) is said to be conservative, for

$$\left(\sum Q_j^{n+1} \, \Delta x - \sum Q_j{}^n \, \Delta x\right)\Big/\Delta t = -(F_+)_{j=J\max} + (F_-)_{j=J\min}, \tag{15}$$

i.e., the time rate of change of the volume integral of Q is equal to the quantity F integrated over the surface enclosing the volume.

Spatial Differencing Errors—Examples

To discuss errors involved in spatial differencing, we take the limit $\Delta t \to 0$, so that semidiscretized equations, i.e., equations discretized in only the spatial dimension, are considered. The semidiscretized form of (6) is

$$\frac{\partial Q_j}{\partial t} + \frac{(F_+)_j - (F_-)_j}{\Delta x} = 0 \tag{16}$$

and (16) is "conservative" if $(F_+)_j = (F_-)_{j+1}$.

If $p = 0$ and $B = 0$, Eqs. (1) and (2) lead to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho v^3\right) = 0, \tag{17}$$

i.e., the differential equations conserve kinetic energy. One conservative method for semidiscretizing (1) and (2) would be

$$\frac{\partial \rho_j}{\partial t} + \frac{(\overline{\rho v})_{j+1/2} - (\overline{\rho v})_{j-1/2}}{\Delta x} = 0, \tag{18}$$

$$\frac{\partial(\rho_j v_j)}{\partial t} + \frac{(\overline{\rho v v})_{j+1/2} - (\overline{\rho v v})_{j-1/2}}{\Delta x} = 0, \tag{19}$$

where the bar indicates an averaging operator, so that

$$\bar{Q}_{j+1/2} = (Q_{j+1} + Q_j)/2. \tag{20}$$

Equations (18) and (19) are correct to second order in Δx and conserve mass and momentum, and, as with their differential analogues, Eqs. (18) and (19) can be combined to give the time derivative of the kinetic energy. To calculate the time derivative, the following identities are needed:

$$(\overline{\rho v v})_{j\pm 1/2} = (\overline{\rho v})_{j\pm 1/2}\,\bar{v}_{j\pm 1/2} + \delta(\rho v)_{j\pm 1/2}\,\delta v_{j\pm 1/2}\,, \tag{21}$$

$$v_j = \bar{v}_{j+1/2} - \delta v_{j+1/2} = \bar{v}_{j-1/2} + \delta v_{j+1/2}\,, \tag{22}$$

where δ is a difference operator

$$\delta Q_{j+1/2} = (Q_{j+1} - Q_j)/2. \tag{23}$$

Combining Eqs. (18) and (19) and using (21) through (23) we find

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_j v_j^2\right) = v_j \frac{\partial}{\partial t}(\rho_j v_j) - \frac{v_j^2}{2}\frac{\partial \rho_j}{\partial t} = -\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} + \frac{q_{j+1/2} + q_{j-1/2}}{\Delta x}, \tag{24}$$

where

$$f_{j\pm 1/2} = \tfrac{1}{2}\overline{\rho v}_{j\pm 1/2}(\bar{v}_{j\pm 1/2})^2 - \tfrac{1}{2}\overline{\rho v}_{j\pm 1/2}(\delta v_{j\pm 1/2})^2 + \bar{v}_{j\pm 1/2}\,\delta v_{j\pm 1/2}\,\delta(\rho v)_{j\pm 1/2} \tag{25}$$

and

$$q_{j\pm 1/2} = \delta(\rho v)_{j\pm 1/2}(\delta v_{j\pm 1/2})^2. \tag{26}$$

Because $\delta Q \approx \frac{1}{2}\Delta x(\partial Q/\partial X)$, the last two terms of (25) are second order and therefore $(f_{j+1/2} - f_{j-1/2})/\Delta x$ can be considered a second-order *conservative* approxima-

tion to the second term of (17). The f's in (24) are fluxes, and the changes in kinetic energy of cell j that they represent appear as opposite charges in an adjacent cell. On the other hand, the q's in (24) are nonconservative, and, in general, even when summed over the entire domain, represent a nonphysical loss or gain of kinetic energy. We note that correct to fourth order

$$(q_{j+1/2} + q_{j-1/2})/\Delta x \approx \frac{1}{4}\Delta x^2 \frac{\partial(\rho v)}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2,$$

which, at least for $\partial(\rho v)/\partial x < 0$, is similar to the von Neumann–Richtmyer [11] "artificial viscosity." If (18) and (19) are used with a conservative approximation to (5), then, for $\partial(\rho v)/\partial x < 0$, the nonconservative terms of (24) indicate a nonphysical viscouslike heating, whereas, for $\partial(\rho v)/\partial x > 0$, they indicate a nonphysical cooling and a corresponding entropy reduction. If an approximation to (3) is used, the nonconservative terms of (24) indicate a loss or gain of energy by the system unless they are specifically accounted for in the thermal energy equation. A result analogous to (24) for two-dimensional incompressible flow has been presented by Arakawa [12], who showed that finite difference approximations to the vorticity equation do not necessarily conserve kinetic energy.

As a second example, consider the fourth term of (2) and the second term of (4). These terms represent the conversion of magnetic energy to kinetic and vice versa and considered together they conserve the sum of kinetic and magnetic energy, i.e.,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{1}{2}B^2\right) + \frac{\partial}{\partial x}(vB^2) = 0. \tag{27}$$

Semidiscretized approximations analogous to (18) and (19) are

$$\frac{\partial(\rho v)}{\partial t} + \frac{\overline{B_+^2} - \overline{B_-^2}}{2\Delta x} = 0, \tag{28}$$

$$\frac{\partial B}{\partial t} + \frac{(\overline{vB})_+ - (\overline{vB})_-}{\Delta x} = 0, \tag{29}$$

where subscripts $j + \frac{1}{2}$ and $j - \frac{1}{2}$ have been replaced by $+$ and $-$, respectively.

Rewriting the second term of (28) as $\bar{B}_+(\delta B_+/\Delta x) + \bar{B}_-(\delta B_-/\Delta x)$, we find

$$\Delta x \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{1}{2}B^2\right) + \bar{v}_+\bar{B}_+\bar{B}_+ - \bar{v}_-\bar{B}_-\bar{B}_- - \delta v_+\,\delta B_+\,\delta B_+ - \delta v_-\,\delta B_-\,\delta B_- = 0. \tag{30}$$

The second and third terms of (30) represent a conservative approximation to the

second term of (27); the last two terms are nonconservative. We note that correct to fourth order $(\delta v_+ \, \delta B_+ \, \delta B_+ + \delta v_- \, \delta B_- \, \delta B_-)/\Delta x$ is an approximation to

$$\tfrac{1}{4}\Delta x^2(\partial v/\partial x)(\partial B/\partial x)^2$$

and, at least for $\partial v/\partial x < 0$, is therefore similar to ohmic dissipation (see Eqs. (3) and (4)) with a resistivity $\eta = -\tfrac{1}{4}\Delta x^2(\partial v/\partial x)$. A nonphysical conversion to/from kinetic and magnetic energy from/to thermal energy or a net loss/gain of energy from the system is indicated by the last two terms of (30).

Spatial Differencing Equations

Consider the following semidiscretized approximations to (1) through (4).

$$\Delta x \frac{\partial \rho}{\partial t} + \tilde{\rho}\tilde{v}_+ \rightarrow \tilde{\rho}_+\tilde{v}_+ = 0, \tag{31}$$

$$\Delta x \frac{\partial \rho \, v}{\partial t} + \bar{v}_+\tilde{\rho}\tilde{v}_+ \rightarrow \bar{v}_-\tilde{\rho}_+\tilde{v}_+ + \delta p_+ + \delta p_- + \bar{B}_+ \, \delta B_+ + \bar{B}_- \, \delta B_- = 0, \tag{32}$$

$$\Delta x \frac{\partial(\rho\epsilon)}{\partial t} + \tilde{\rho}\tilde{v}\tilde{\epsilon}_+ \rightarrow \tilde{\rho}_+\tilde{v}_+\tilde{\epsilon}_+ + \bar{p}_+ \, \delta v_+ + \bar{p}_- \, \delta v_- - 2\tilde{\eta}_+ \frac{(\delta B_+)^2}{\Delta x} - 2\tilde{\eta}_- \frac{(\delta B_-)^2}{\Delta x} = 0, \tag{33}$$

$$\Delta x \frac{\partial B}{\partial t} + \bar{v}_+\bar{B}_+ - \bar{v}_-\bar{B}_- - 2\tilde{\eta}_+ \frac{\delta B_+}{\Delta x} + 2\tilde{\eta}_- \frac{\delta B_-}{\Delta x} = 0, \tag{34}$$

where again the subscripts $j + \frac{1}{2}$ and $j - \frac{1}{2}$ are replaced by $+$ and $-$, respectively. Without any specification of the operation indicated by the tilde (~), Eqs. (31)–(34) conserve mass, momentum, energy, and magnetic flux. In addition, Eqs. (31)–(34) have the following desirable "subconservation" properties which are not normally found in difference methods which automatically guarantee energy conservation by differencing Eq. (5) in a conservative form.

(a) the approximations to $\partial(\rho v)/\partial x$ and $\partial(\rho vv)/\partial x$ conserve kinetic energy; the kinetic energy flux is $\tilde{\rho}\tilde{(}\bar{v}^2 - \delta v^2)/2$;

(b) the approximations to $\partial p/\partial x$ and $p(\partial v/\partial x)$ conserve the sum of kinetic and thermal energies; the energy flux is $\bar{p}\bar{v}$;

(c) the approximations to $B(\partial B/\partial x)$ and $\partial(vB)/\partial x$ conserve the sum of kinetic and magnetic energies; the energy flux is $\bar{v}\bar{B}\bar{B} - \bar{B} \, \delta v \, \delta B$;

(d) the approximation to $\eta(\partial B/\partial x)^2$ and $(\partial/\partial x) \, \eta(\partial B/\partial x)$ conserve the

sum of thermal and magnetic energies; the energy flux is $-\tilde{\eta}\bar{B}(\delta B/\Delta x)$. Where the averaging operator $(^{-})$ and the differencing operator δ are indicated they are specifically required to maintain the properties (a)–(d) listed above. The tilde $(^{\sim})$ indicates terms which can be treated somewhat arbitrarily without destroying the energy conservation properties; these terms, for example, can be determined from stability considerations. Unless (33) is replaced by an equation automatically guaranteeing energy conservation regardless of the form of (31), (32), and (34), any variation from (31), (32), and (34) requires an accompanying change in the internal energy equation (33) to assure energy conservation. A likely addition to (32) is, of course, an "artificial viscosity" such as used by von Neumann and Richtmyer. If an equation of the form (5) is used, energy conservation is still guaranteed if an "artificial viscosity" is used, but it would appear that a modificiation of the energy flux should still be incorporated.

The various terms appearing in Eqs. (1) through (4) are representative of the terms appearing in multidimensional magnetohydrodynamic partial differential equations. Even when the equations are written in curvilinear coordinate systems, the analysis presented above of conservation properties of the spatial difference equations can be extended in a straightforward manner; the scale factors which appear in curvilinear coordinates increase the arbitrariness indicated in (31)–(34) by the tilde $(^{\sim})$. The difference equations can be thus cast in a form which, in the limit $\Delta t \to 0$, assure conservation of mass, momentum, energy, and flux and in addition maintain all of the "subconservation" properties analogous to properties (a)–(d) discussed above, even though an internal energy equation analogous to (3) is used. Thus, as with their differential analogs, finite difference approximations to both conservation and nonconservation forms of the energy transport equation can, at least in the limit $\Delta t \to 0$, be equivalent.

Temporal Differencing Errors—Example

Using the spatial differencing indicated above we consider explicit finite difference approximations to Eqs. (3), (4), and (5) with the velocity set to zero. For (4) and (5) we have

$$\frac{B^{+} - B}{\Delta t} - 2\,\frac{\tilde{\eta}_{+}\,\delta B_{+} - \tilde{\eta}_{-}\,\delta B_{-}}{(\Delta x)^2} = 0, \tag{35}$$

$$\frac{\rho^{+}\epsilon^{+} - \rho\epsilon}{\Delta t} + \frac{B^{+}B^{+} - BB}{2\Delta t} - 2\,\frac{\tilde{\eta}_{+}\bar{B}_{+}\,\delta B_{+} - \tilde{\eta}_{-}\bar{B}_{-}\,\delta B_{-}}{(\Delta x)^2} = 0, \tag{36}$$

respectively. In (35) and (36) the superscript $+$ indicates values at time t^{n+1} and all other values are at time t^{n}; in addition, unsubscripted quantities are values at x_j and

the subscripts $+$ and $-$ replace $j+\frac{1}{2}$ and $j-\frac{1}{2}$, as before. Using the identity $(B^+)^2 - B^2 = 2B(B^+ - B) + (B^+ - B)^2$ and using (35), we find from (36)

$$\frac{\rho^+\epsilon^+ - \rho\epsilon}{\Delta t} - \frac{2\tilde{\eta}_+(\delta B_+)^2}{(\Delta x)^2} - \frac{2\tilde{\eta}_-(\delta B_-)^2}{(\Delta x)^2} + \frac{(B^+ - B)^2}{2\Delta t} = 0. \tag{37}$$

A finite difference approximation to (3) would have all terms of (37) except the fourth. Since the fourth term is always positive, it represents a nonphysical cooling resulting from the automatic conservation of energy guaranteed by Eq. (36). Had an implicit formulation been used in (35) and (36) the sign of the fourth term of (37) would have been negative; i.e., an implicit conservation formulation results in nonphysical heating. We note, however, that a Crank–Nicholson-like method ($B \to \hat{B} = (B^+ + B)/2$ in the spatial difference equations) eliminates the nonphysical term.

Temporal Difference Equations

To demonstrate in general terms the errors introduced by conservative differencing of (5), we write difference approximations to (1) through (5) as follows.

$$[(\rho^+ - \rho)/\Delta t] + F_1^+ + G_1 = 0, \tag{38}$$

$$[(\rho^+ v^+ - \rho v)/\Delta t] + F_2^+ + G_2 = 0, \tag{39}$$

$$[(\rho^+\epsilon^+ - \rho\epsilon)/\Delta t] + F_3^+ + G_3 = 0, \tag{40}$$

$$[(B^+ - B)/\Delta t] + F_4^+ + G_4 = 0, \tag{41}$$

$$\frac{\rho^+\epsilon^+ - \rho\epsilon}{\Delta t} + \frac{\rho^+ v^+ v^+ - \rho v v}{2\Delta t} + \frac{B^+B^+ - B^2}{2\Delta t} + F_5^+ + G_5 = 0, \tag{42}$$

where the F's are implicit quantities and the G's are explicit quantities. The F's and G's are spatially differenced quantities, and for (40) and (42) to be equivalent in the limit $\Delta t \to 0$, we require

$$F_3^+ + B^+F_4^+ + v^+F_2^+ - [(v^+)^2/2]\,F_1^+ = F_5^+, \tag{43}$$

$$G_3 + BG_4 + vG_2 - (v^2/2)\,G_1 = G_5\,. \tag{44}$$

From (41) we obtain

$$\frac{B^+B^+ - B^2}{2\Delta t} = \frac{B^+ + B}{2}\frac{B^+ - B}{\Delta t} = -B^+F_4^+ - BG_4 + \frac{(B^+ - B)}{2}(F_4^+ - G_4), \tag{45}$$

and from (38) and (39)

$$\frac{\rho^+v^+v^+ - \rho vv}{2\Delta t} = \frac{v + v^+}{2}\frac{(\rho^+v^+ - \rho v)}{\Delta t} - \frac{vv^+}{2}\frac{\rho^+ - \rho}{\Delta t}$$

$$= -v^+F_2^+ - vG_2 + \frac{v^+v^+F_1^+}{2} + \frac{vv}{2}G_1$$

$$+ \frac{(v^+ - v)}{2}(vG_1 - G_2 - v^+F_1^+ + F_2^+). \tag{46}$$

Substituting (43) through (46) into (42) we find

$$\frac{\rho^+\epsilon^+ - \rho\epsilon}{\Delta t} + F_3^+ + G_3 + \frac{(v^+ - v)}{2}(vG_1 - G_2 - v^+F_1^+ + F_2^+)$$

$$+ \frac{(B^+ - B)}{2}(F_4^+ - G_4) = 0. \tag{47}$$

The last two terms on the left-hand side of (47) represent the nonphysical heating/cooling which results from temporal differencing; or, they indicate the energy loss/gain which would result if Eq. (40) were used instead of (42). The error terms are $0(\Delta t)$ because of the time differences $v^+ - v$ and $B^+ - B$ and hence become smaller as the time step is reduced. The error can be reduced to $0(\Delta t^2)$ if the spatial derivatives are written as a time averages so that $G = F(t^n)$ where $F^+ = F(t^{n+1})$. The error is then

$$\frac{1}{2}\Delta t^2\left[\frac{\partial v}{\partial t}\left(\frac{\partial F_2}{\partial t} - \frac{\partial(vF_1)}{\partial t}\right) + \frac{\partial B}{\partial t}\frac{\partial F_4}{\partial t} + 0(\Delta t^2)\right];$$

the error discussed here is only the nonphysical heating/cooling, not, of course, the full truncation error.

Most of the nonphysical terms which appear in (47) are removed if Crank–Nicholson-like differencing is used. If all G's are zero and the F's satisfy

$$F_3^+ + \frac{B^+ + B}{2}F_4^+ + \frac{v^+ + v}{2}F_2^+ - \frac{(v^+ + v)^2}{8}F_1^+ = F_5^+, \tag{48}$$

so that (40) and (42) are equivalent, we find from (44), using the first equalities of (45) and (46),

$$[(\rho^+\epsilon^+ - \rho\epsilon)/\Delta t] + F_3^+ - [(v^+ - v)^2/8]F_1^+ = 0. \tag{49}$$

The last term of (49) is nonphysical but is $0(\Delta t^2)$.

The second order (in Δt) methods discussed above would appear to be desirable from an accuracy point of view, but they are implicit, and solution of fully implicit multidimensional difference equations are well known to be sufficiently complicated to prohibit their use in most cases. Lindemuth and Killeen [13] have reported using an alternating-direction-implicit (ADI) [14] method for solving time-dependent two-fluid two-dimensional magnetohydrodynamic equations which alternately uses implicit and explicit finite difference equations for the spatial derivatives in each of the two dimensions. Such an alternating direction procedure is considerably less complicated than a fully implicit two-dimensional method. An alternating-direction-implicit approximation reduces in the one-dimensional case to the use of the exact same values for the spatial derivatives on two consecutive time steps, i.e., for (1),

$$[(\rho_j^{n+1} - \rho_j^n)/\Delta t] + F_1^{n+1} = 0, \tag{50}$$

$$[(\rho_j^{n+2} - \rho_j^{n+1})/\Delta t] + F_1^{n+1} = 0. \tag{51}$$

In (50) F^{n+1} is implicit because it involves unknown quantities at the new time t^{n+1}, whereas in (51) it is explicit because it involves known quantities at the old time t^{n+1}. With equations analogous to (50) and (51) for (2), (4), and (5), we find, using (47), that an equation guaranteeing *total* energy conservation leads to

$$\frac{\rho^{n+2}\epsilon^{n+2} - \rho^n\epsilon^n}{2\Delta t} + F_3^{n+1} + \frac{(v^{n+1}F_1^{n+1} - F_2^{n+1})}{4}(v^{n+2} - 2v^{n+1} + v^n) = 0. \tag{52}$$

The last term in (52) represents a nonphysical heating or cooling. Because $v^{n+2} - 2v^{n+1} + v^n$ is an approximation to $\Delta t^2(\partial^2 v/\partial t^2)$, the error in (52) is second order. The result (52) extends directly to ADI methods in two dimensions. When an equation analogous to (42) guaranteeing total energy conservation is used, alternating-direction-implicit time differencing used with appropriate spatial differencing will lead to a second-order nonphysical heating or cooling. On the other hand, if an equation analogous to (40) is used, a second-order (in Δt) loss or gain of energy from the system will occur. The $0(\Delta t)$ errors introduced in ADI calculations on one time step are approximately canceled (some are exactly canceled) on the following time step.

Equations (47), (49), and (52) indicate that the conservation errors due to time differencing cannot be fully eliminated. If an equation analogous to (40) is used, energy conservation checks are useful to indicate whether or not the time step has been maintained at a sufficiently low value to obtain the required accuracy even when numerical stability considerations would have allowed a larger time step.

Summary and Concluding Remarks

Errors in the numerical solution of magnetohydrodynamics equations can be divided into two classes. The first class is the well-documented stability, dispersion, and diffusion errors which are present even when the equations reduce to linear ones and which lead to the nonphysical transport of the various quantities. This paper has discussed a second class of errors, the errors which result from the nonlinearity of the equations and which lead to nonphysical losses and gains of energy, or interchanges from one form of energy to another. The sources of this second class of errors have been shown to be both the spatial and temporal finite difference methods used. Without completely specifying a set of difference equations, this paper has shown that the second class can be eliminated only in the limit $\Delta t \to 0$. In addition it has been shown that appropriately chosen implicit or alternating-direction-implicit methods, in addition to their well-known enhanced numerical stability, can reduce the second class of errors to second order in the time step Δt.

The analysis presented in this paper indicates why, for most finite difference methods which have been implemented in magnetohydrodynamics, the approximations to the conservation and nonconservation forms of the internal energy equation are not equivalent even though their differential analogues are equivalent. Because of the possible existence of nonphysical energy interchange mechanisms, it would appear that the use of methods whose accuracy has been proven in conservation form, such as Lax–Wendroff, Lapidus smoothing [15], flux-corrected transport [16], to name a few, should be done cautiously if a nonconservation form of the internal energy equation is to be used; necessary mechanisms such as "artificial viscosity" (and perhaps even "artificial resistivity") should be identified and undesirable mechanisms, such as those which reduce entropy, should be removed.

It is difficult to estimate the magnitude of the errors discussed here, and, of course, the errors become smaller as the finite difference space–time mesh is refined. We have incorporated most of the ideas presented above in the computer code ANIMAL—*A* *N*ew (Alternating Direction) *I*mplicit *M*agnetohydrodynamic *AL*gorithm—which also uses some of the techniques reported earlier [13, 17]. The incorporation of the ideas presented here has given ANIMAL an improved capability to calculate accurately a wide class of problems. Because ANIMAL maintains the "subconservation" properties introduced above, we feel we have a code which obtains numerical solutions which retain more of the properties of the magnetohydrodynamic partial differential equations than any previously reported code.

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